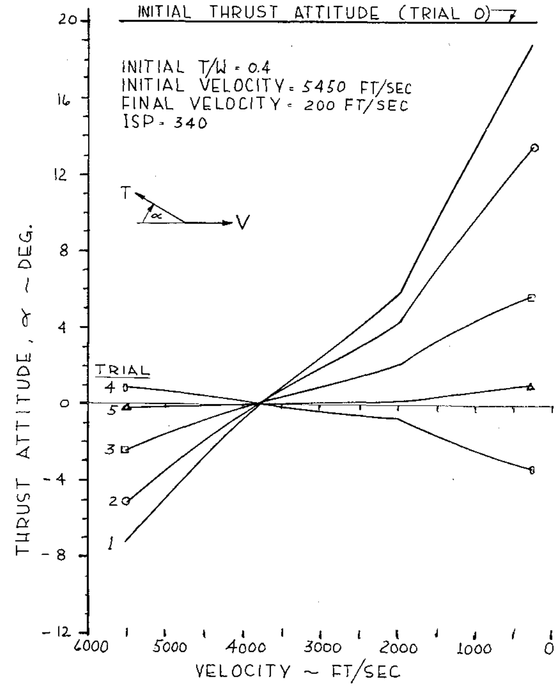

Fig. 1 Thrust attitude on successive optimization trials

Table 1 Pay-off and lateral deviation on successive trials

Trial	Mass ratio / Optimum mass ratio	Lateral deviation
0	1.03125	165,000
1	1.00497	−806
2	1.00255	67
3	1.00050	90
4	1.00014	41
5	1.00001	−6

course, is that the vehicle should move on a straight line with the thrust pointed in a direction opposite to the velocity vector. The first guess at the thrust attitude, however, is 20° away from this direction. Figure 1 illustrates the thrust attitude histories along successive trials. The discontinuity in slope at a velocity of 2000 fps is due to the fact that the guidance parameters are not recomputed beyond this point. It is not possible to continue the closed-loop calculations to the end of the trajectory, because excessively large control changes would be called for to correct small errors in the terminal quantity. In Table 1, the improvement in pay-off and the error in the terminal constraint are shown. It is seen that most of the improvement is accomplished on the first trial by just coming close to meeting the terminal constraint. After three trials the terminal mass is quite close to the optimum, although the thrust attitude is as much as 5° away from the optimum. Further improvement in thrust attitude produces very little change in terminal mass.

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Shearing Flow of a Viscoelastic Fluid between Porous Coaxial Cylinders

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THE problem of a Couette-type flow of a viscoelastic fluid between parallel walls, of which the fixed one is porous, recently has been investigated by the author.[1] In the present note, the related problem of shearing flow of a viscoelastic fluid between two coaxial cylinders, in which the outer one is moving parallel to its axis and the inner fixed cylinder is porous, is considered. This problem in the case of viscous fluid has been discussed by Dunwoody.[2]

The stress-strain relations for an incompressible viscoelastic fluid are given as

$$\sigma_j^i + \tau\tilde{\sigma}_j^i = 2\mu e_j^i \tag{1}$$

where σ_j^i is the extra-stress tensor, τ the elastic constant, μ the coefficient of viscosity, and e_j^i the rate of strain tensor. The term $\tilde{\sigma}_j^i$ appearing in Eq. (1) denotes its rate of change, which, following Truesdell,[3] one takes as

$$\tilde{\sigma}_j^i = (\partial\sigma_j^i/\partial t) + \sigma_{j,k}^i v^k + \sigma_j^i v_{,k}^k - \sigma^{ik}v_{j,k} - \sigma_j^k v_{,k}^i \tag{2}$$

Measuring z coordinate along the common axis of the cylinders and assuming axial symmetry, the equations of motion and continuity governing the problem are

$$\rho\left[u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial r} + \frac{\partial\sigma_r^r}{\partial r} + \frac{\partial\sigma_z^r}{\partial z} + \frac{\sigma_r^r - \sigma_\theta^\theta}{r} \tag{3}$$

$$\rho\left[u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \frac{\partial\sigma_z^r}{\partial r} + \frac{\partial\sigma_z^z}{\partial z} + \frac{\sigma_z^r}{r} \tag{4}$$

and

$$(\partial u/\partial r) + (u/r) + (\partial w/\partial z) = 0 \tag{5}$$

respectively.

If one further assumes the velocities to be functions of r only, Eqs. (3–5) reduce to

$$\rho u\frac{du}{dr} = -\frac{\partial p}{\partial r} + \frac{d\sigma_r^r}{dr} + \frac{\sigma_r^r - \sigma_\theta^\theta}{r} \tag{6}$$

$$\rho u\frac{dw}{dr} = -\frac{\partial p}{\partial z} + \frac{d\sigma_z^r}{dr} + \frac{\sigma_z^r}{r} \tag{7}$$

$$(1/r)(d/dr)(ru) = 0 \tag{8}$$

Also, Eq. (1), giving the stress-strain relation, becomes

$$\sigma_r^r + \tau\left[u\frac{d\sigma_r^r}{dr} - 2\sigma_r^r\frac{du}{dr}\right] = 2\mu\frac{du}{dr} \tag{9}$$

$$\sigma_z^r + \tau\left[u\frac{d\sigma_z^r}{dr} - \sigma_r^r\frac{dw}{dr} - \sigma_z^r\frac{du}{dr}\right] = \mu\frac{dw}{dr} \tag{10}$$

$$\sigma_z^z + \tau\left[u\frac{d\sigma_z^z}{dr} - 2\sigma_z^r\frac{dw}{dr}\right] = 0 \tag{11}$$

The boundary conditions of the problem are

$$u = U_0 \qquad w = 0 \text{ at } r = R_1 \tag{12}$$
$$w = U \text{ at } r = R_2$$

From Eqs. (8) and (12), one has $ru = \text{const} = R_1U_0$. This

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shows that the rate of withdrawal at the outer cylinder must be the same as the rate of injection at the inner cylinder and vice-versa. Also from Eq. (6), $\partial^2 p/\partial r\partial z = 0$, whereas Eq. (7) shows that $\partial p/\partial z$ is a function of r only. Hence $\partial p/\partial z$ must be constant. Eliminating $\sigma_r^{\,r}$ between Eqs. (9) and (10) and retaining the terms to the first order of τ, one gets

$$\sigma_z^{\,r} + \tau\left[u\frac{d\sigma_z^{\,r}}{dr} - 2\mu\left(\frac{du}{dr}\right)\left(\frac{dw}{dr}\right) - \sigma_z^{\,r}\frac{du}{dr}\right] = \mu\frac{dw}{dr}$$

which, on putting $u = R_1U_0/r$, gives

$$\sigma_z^{\,r}\left(1 + \frac{\tau R_1 U_0}{r^2}\right) + \frac{\tau R_1 U_0}{r}\frac{d\sigma_z^{\,r}}{dr} = \mu\frac{dw}{dr}\left(1 - \frac{2\tau R_1 U_0}{r^2}\right) \quad (13)$$

Substituting the value of u in Eq. (7) and integrating, one has

$$C + w\rho R_1 U_0 + (r^2/2)(\partial p/\partial z) = r\sigma_z^{\,r} \quad (14)$$

where C is the constant of integration.

Eliminating $\sigma_z^{\,r}$ between Eqs. (13) and (14) and simplifying, one gets

$$\frac{1}{r}\frac{dw}{dr}\left[r^2 - \tau\left(2R_1U_0 + \frac{R_1U_0^2}{\nu}\right)\right] - \frac{wR_1U_0}{\nu} - \frac{1}{\mu}\frac{\partial p}{\partial z}\left(\frac{r^2}{2} + \tau R_1 U_0\right) - \frac{c}{\mu} = 0 \quad (15)$$

Substituting $w = \bar{w}U$, $r = R_1\bar{r}$, $z = R_1\bar{z}$, $Re = UR_1/\nu$, $Rw = U_0R_1/\nu$, $P = -(1/\mu U)(\partial p/\partial\bar{z})$, $k = \tau[(2U_0/R_1) + (U_0^2/\nu)]$, and $k' = \tau U_0$ in Eq. (15), one obtains

$$\frac{1}{\bar{r}}\frac{d\bar{w}}{d\bar{r}}(\bar{r}^2 - k) - Rw\,\bar{w} + P\left(\frac{R_1\bar{r}^2}{2} + k'\right) - \frac{C}{\mu U} = 0 \quad (16)$$

with the boundary conditions

$$\bar{w} = 0 \qquad \bar{r} = 1 \qquad \bar{w} = 1 \qquad \bar{r} = R_2/R_1 = \beta \quad (17)$$

The solution of Eq. (16) is

$$\bar{w} = A + B(r^2 - k)^{Rw/2} - \frac{PR_1(r^2 - k)}{2(2 - Rw)} + \frac{P}{2Rw}(R_1k + 2k') \quad (18)$$

where A and B are arbitrary constants to be determined from Eq. (17). There, values are

$$A = \frac{1}{[(\beta^2 - k)^{Rw/2} - (1 - k)^{Rw/2}]} \times \left\{\frac{PR_1[(\beta^2 - k)^{Rw/2}(1 - k) - (\beta^2 - k)(1 - k)^{Rw/2}]}{2(2 - Rw)} - (1 - k)^{Rw/2}\right\} - \frac{P}{2Rw}(R_1k + 2k') \quad (19)$$

$$B = \frac{1}{[(\beta^2 - k)^{Rw/2} - (1 - k)^{Rw/2}]}\left[1 + \frac{PR_1(\beta^2 - 1)}{2(2 - Rw)}\right] \quad (20)$$

From Eq. (18), it is interesting to observe that a change in the velocity profile appears due to the presence of suction or injection, even in the absence of elasticity. But the effect of elasticity is perceptable only when the cylinders are porous. (It may be verified from the fact that τ always is coupled with U_0.) This is in agreement with the analogous result of Ref. 1.

At the fixed cylinder, separation occurs when $(d\bar{w}/d\bar{r})_{\bar{r}=1} = 0$, which from Eqs. (18) and (20) is equivalent to

$$PR_1 = \frac{2Rw(2 - Rw)(1 - k)^{-[1-(Rw/2)]}}{[2\{\beta^{Rw}[1 - (k/\beta^2)^{Rw/2}] - (1 - k)^{Rw/2}\} - Rw(\beta^2 - 1)(1 - k)^{-[1-(Rw/2)]}]} \quad (21)$$

If one puts $\beta^{Rw} = \exp(Rw\log\beta)$ and expands the exponential and binomials, then for $|Rw| \ll 1$, Eq. (21) can be written as

$$\frac{1}{(\frac{1}{2}\rho U^2)}\frac{\partial p}{\partial\bar{z}} = \frac{4(1 + k)}{Re\{(\beta^2 - 1) - 2\log\beta + k[(\beta^2 - 1)/\beta]^2\}}\left\{2 - Rw\left[\frac{1 + 2k}{1 + k} - \frac{2[(\log\beta)^2 - k\{(\log\beta/\beta^2) + [(\beta^2 - 1)/2]\}]}{\{(\beta^2 - 1) - 2\log\beta + k[(\beta^2 - 1)/\beta]^2\}}\right] + O(Rw^2)\right\} \quad (22)$$

where $\{(\beta^2 - 1) - 2\log\beta + K[(\beta^2 - 1)/\beta]^2\}$ and the term in Rw are positive-definite functions of parameter $\beta > 1$.

From Eq. (22) it is observed that, for blowing ($Rw > 0$), a smaller adverse pressure gradient than for sucking ($Rw < 0$) will cause the flow to separate. This result is analogous to that of Ref. 2. However the point that seems to be of physical interest is that the existence of elasticity quickens the separation.

The effect of β, the radius ratio, on the pressure gradient given by Eq. (22) can be discussed as follows. Write Eq. (22) as

$$\frac{1}{\frac{1}{2}\rho U^2}\frac{\partial p}{\partial\bar{z}} = \frac{4}{Re}\left\{\frac{2}{(\beta^2 - 1) - 2\log\beta + k[(\beta^2 - 1)/\beta]^2} - Rw\left[\frac{[(1 + 2k)/(1 + k)]\{(\beta^2 - 1) - 2\log\beta + k[(\beta^2 - 1)/\beta]^2\} - 2(\log\rho)^2 + 2k\{(\log\beta/\beta^2) + [(\beta^2 - 1)/2]\}}{\{(\beta^2 - 1) - 2\log\beta + k[(\beta^2 - 1)/\beta]^2\}^2} + O(Rw)^2\right]\right\} \quad (23)$$

and discuss the two limiting cases of parameter β.

Case A: $\beta \to \infty$

In this case, both terms on the right-hand side of Eq. (23) tend to zero in the limit, and therefore the adverse pressure gradient vanishes.

Case B: $\beta \to 1$

Putting $\beta = 1 + \epsilon$ and expanding the right-hand side of Eq. (25), one obtains (preserving the significant powers of ϵ)

$$\frac{1}{(\frac{1}{2}\rho U^2)}\frac{\partial p}{\partial\bar{z}} = \frac{4}{Re}\left\{\frac{1}{\epsilon^2 + (k/2)\{\epsilon + [\epsilon/(1 + \epsilon)]\}^2} - Rw\left[\frac{1}{3\epsilon\{1 + (k/2)[1 + (1/\epsilon)]^2\}^2} + kf(\epsilon)\right]\right\} \quad (24)$$

where $f(\epsilon)$ is a positive function of the variable ϵ of orders $1/\epsilon$ and lower. It is clear from Eq. (24) that, as $\epsilon \to 0$, the adverse pressure gradient becomes increasingly large.

The conclusion that comes from these results is that, the smaller the gap between the cylinders, the greater will be the adverse pressure gradient to provoke separation. This result is identical to that observed in Ref. 2. The effect of the elasticity of the fluid is, however, to accelerate the separation.

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Minimum Impulse Orbital Transfers

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SOLUTIONS have been obtained[1–3] for the absolute optimum two-impulse orbital transfer from a circular or apsidal terminal to an outer coplanar orbit, as well as from an inner orbit to an outer coplanar circular or apsidal terminal. The purpose of this paper is to extend these results so as to obtain the absolute optimum two-impulse transfer from an arbitrary inner terminal to an outer coplanar orbit and from an inner orbit to an arbitrary outer coplanar terminal. As in Refs. 2 and 3, terminal implies a radial distance and a velocity vector in an inverse square gravitational field.

Throughout this paper, the following idealizing assumptions have been made:

1) All given orbits are closed coplanar orbits having the same rotational sense.
2) The two orbits connected by the transfer orbit are such that they cannot intersect regardless of their relative orientation.
3) All velocity changes arise from instantaneous impulses.
4) The attracting body and the orbiting vehicle are constant point-masses in an idealized two-body system.
5) Transfers requiring more than two impulses are not considered.

The two orbits connected by the transfer orbit are called the inner and outer orbits, the inner orbit being that orbit with the smaller perigee altitude. Outer terminal and inner terminal refer, respectively, to terminals lying on the outer and inner orbits.

Of particular importance in the following derivations is a result of Ref. 4. There it was shown that, under the foregoing assumptions, the total impulse requirement ΔV_T for the absolute optimum two-impulse transfer from an arbitrary terminal whose radial distance is R_1 and whose velocity components are (u_0, v_0) to an arbitrary outer terminal whose radial distance is R_2 and whose velocity components are (u_F, v_F) is

$$\Delta V_T = [\{u_F + [2GMR_2/R_1(R_1 + R_2)]^{1/2}\}^2 + v_F^2]^{1/2} - [\{u_0 + [2GMR_1/R_2(R_1 + R_2)]^{1/2}\}^2 + v_0^2]^{1/2} \quad (1)$$

where G and M are, respectively, the gravitational constant and the mass of the attracting body.

Transfer from Inner Terminal to Outer Orbit

Because of the forementioned result of Ref. 4, the problem in this case becomes to vary the final terminal over the outer orbit, always making the optimum transfer [given by Eq. (1)] to the variable terminal, and to determine the location of the final terminal so that the outer orbit is achieved most economically.

It is mathematically convenient to express the total impulse requirement for the optimum terminal-to-terminal transfer in dimensionless multiples of circular velocity C_1 at the radial distance of the fixed initial terminal. Eq. (1) then becomes

$$\Delta V_T/C_1 = [\{\eta_F + [2/(1 + \rho)]^{1/2}\}^2 + \xi_F^2]^{1/2} - [\{\eta_0 + \rho[2/(1 + \rho)]^{1/2}\}^2 + \xi_0^2]^{1/2} \quad (2)$$

where

$$(u_i, v_i) = (C_1\eta_i, C_1\xi_i) \quad (3)$$

and

$$\rho = R_1/R_2 \quad (4)$$

If R_p and R_a are, respectively, the radial distances of the perigee and apogee of the orbit containing the outer terminal, the dimensionless perigee velocity η_p on the outer orbit is given by

$$\eta_p^2 = 2R_1R_a/R_p(R_a + R_p) \quad (5)$$

Since the rotational sense of the two orbits is the same, η_0, η_F, and η_p have the same sign, which is taken to be positive.

From the law of conservation of angular momentum,

$$\eta_F = \kappa\rho \quad (6)$$

where

$$\kappa = R_p\eta_p/R_1 = \text{const} \quad (7)$$

Combining the laws of conservation of angular momentum and total energy yields

$$\xi_F^2 = (\eta_p - 1/\kappa)^2 - (\kappa\rho - 1/\kappa)^2 \quad (8)$$

With Eqs. (6) and (8), $\Delta V_T/C_1$ is a continuous function of a single variable ρ in the closed interval

$$0 < R_1/R_a \le \rho \le R_1/R_p < 1 \quad (9)$$

Since neither term in Eq. (2) can vanish, $\Delta V_T/C_1$ has a continuous derivative in the interval of the variable ρ. Differentiating and simplifying gives

$$\frac{d(\Delta V_T/C_1)}{d\rho} = \frac{2^{-1/2}(2+\rho)}{(1+\rho)^{3/2}}\left\{\frac{\kappa + \rho[2/(1+\rho)]^{1/2}}{[\{\eta_F + [2/(1+\rho)]^{1/2}\}^2 + \xi_F^2]^{1/2}} - \frac{\eta_0 + \rho[2/(1+\rho)]^{1/2}}{[\{\eta_0 + \rho[2/(1+\rho)]^{1/2}\}^2 + \xi_0^2]^{1/2}}\right\} \quad (10)$$

The second fraction inside the braces of Eq. (10) is always ≤ 1 so that $\Delta V_T/C_1$ has a positive first derivative when

$$\kappa + \rho[2/(1+\rho)]^{1/2} > [\{\eta_F + [2/(1+\rho)]^{1/2}\}^2 + \xi_F^2]^{1/2} \quad (11)$$

Assuming that Eq. (11) does not hold and using Eqs. (5–8) gives

$$1 \le R_1/R_a \quad (12)$$

contradicting Eq. (9).

Thus, $\Delta V_T/C_1$ has a positive first derivative at every point in the interval, and the absolute optimum two-impulse transfer from the arbitrary inner terminal to the outer coplanar orbit is a transfer to the apogee of the outer orbit.

Using these results in Eq. (1) shows that the total impulse requirement for the optimum transfer is

$$(\Delta V_T)_{\min} = u_a + [2GMR_a/R_1(R_1 + R_a)]^{1/2} - [\{u_0 + [2GMR_1/R_a(R_1 + R_a)]^{1/2}\}^2 + v_0^2]^{1/2} \quad (13)$$

where u_a is the apogee velocity on the outer orbit.

From the results of Ref. 4 it can be shown that the apogee of the transfer orbit coincides with the apogee of the outer orbit, and that the apogee velocity u_2 on the optimum transfer orbit is

$$u_2 = C_1\sigma\left(\frac{2}{1+\sigma}\right)^{1/2}\left[\frac{1 - \sigma(1 + B^2)^{1/2}}{(1 + B^2)^{1/2} - \sigma}\right] \quad (14)$$

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